Local Factors and the Plancherel Measure

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Overview

(k local, G/k reductive)

smooth irreps

$$(WD_k \xrightarrow{\varphi} {}^LG,\ \rho \in \mathrm{Irr}(S_\varphi)) \xleftrightarrow[\text{1-1}]{\text{LLC}} \pi_{(\varphi,\rho)} \in \mathrm{Irr}(G)$$

$\rightsquigarrow$ L- and ε-factors (part I)

Q: What do local factors tell us about $\mathrm{Irr}(G)$?

Rough answer:

- $\mathrm{sgn}(\varepsilon)$: central character at involution
- $\underbrace{L + \varepsilon}_{\gamma}$: Plancherel measure (parts II + III)

I. Local Factors

Definition of local factors

W-D reps $(V/\mathbb{C})$

unramified data

$$\phi \mapsto L(s,\phi) = \det\left(1 - q^{-s}\,\phi(\mathrm{Fr})\,\middle|\,V_N^{I}\right)^{-1}$$

$$WD_k \xrightarrow{\ \phi\ } GL(V)$$

mysterious sign

$$\phi \mapsto \varepsilon(s,\phi) = \underbrace{\varepsilon(\tfrac{1}{2},\varphi)}_{\text{root number}} \cdot q^{(\frac{1}{2}-s)\,\mathrm{cond}(\phi)}$$

(Deligne '72)

ramified data

L-params

$(q=1$ for k arch.$)$

$$\begin{tikzcd} WD_k \arrow[r, "\varphi"] \arrow[rr, bend right] & {}^{L}G \arrow[r, "r"] & GL(V) \end{tikzcd}$$

$$\mapsto L(s,\varphi,r) = L(s, r\circ\varphi) \qquad \text{(part II)}$$

$$\mapsto \varepsilon(s,\varphi,r) = \varepsilon(s, r\circ\varphi) \qquad \text{(part I)}$$

Root numbers trichotomy

$$WD_k \xrightarrow{\phi} GL(V)$$

$$|\varepsilon(\tfrac{1}{2},\phi)|=1$$

$\phi^\vee \not\simeq \phi$: $\varepsilon(\tfrac{1}{2},\phi)$ randomly distributed.

(cf. Kummer sums, Heath-Brown + Patterson)

$\phi^\vee \simeq \phi$: $\varepsilon(\tfrac{1}{2},\phi)^2 = \det(\phi)(-1) = \pm 1$.

- Symplectic : ??? (GGP)
- Orthogonal : Spin lifting

$$\begin{tikzcd} & \mathrm{Spin}(V) \arrow[d] \\ WD_k \arrow[ur, dashed, "\exists ?"] \arrow[r, "\phi"] & SO(V) \end{tikzcd}$$

Orthogonal root numbers of tempered parameters

Thm: (S) $WD_k \xrightarrow[\text{tempered}]{\varphi} {}^L G \xrightarrow[\text{orthogonal}]{r} O(V)$. Then

$$\varepsilon\left(\tfrac{1}{2}, \varphi, r\right) = \varepsilon\left(\tfrac{1}{2}, \varphi_{prin}, r\right) \cdot \chi_\varphi(z_r).$$

Here

φ_{prin}: $\left(WD_k \twoheadrightarrow SL_2(\mathbb{C}) \longrightarrow \hat{G}\right)$ (1 for k arch.)

$\chi_\varphi : Z(G) \longrightarrow \mathbb{C}^\times$ central. char. of L-packet

$$z_r := \prod_{0 < \varpi \in X^*(\hat{T})} \varpi(-1)^{m(r,\varpi)} \in Z(G) \quad (= \pm 1)$$

II. Plancherel Measure

Tempered dual

Levi — supercuspidal of M — unramified char. — only for k nonarch.

discrete series of M

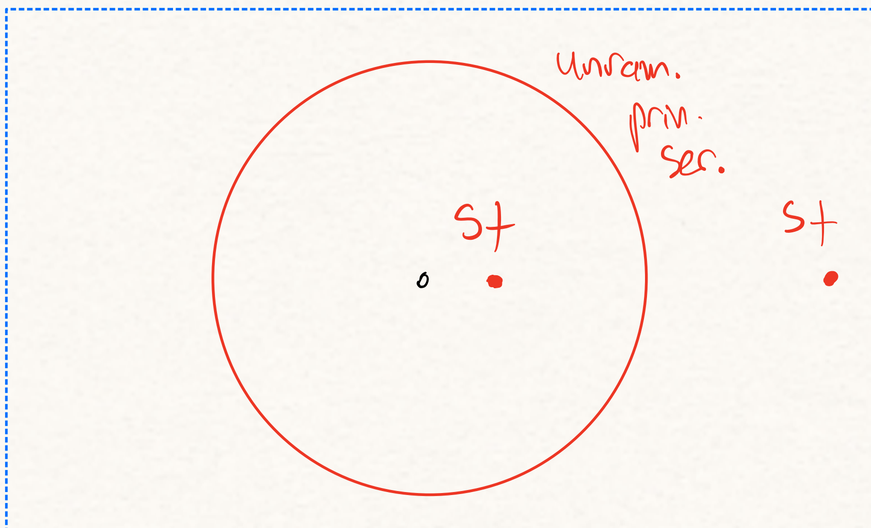
$$\begin{tikzcd}
\mathrm{Irr}(G) \arrow[r, two heads, "sc"] & \{(M, \omega_0) \curvearrowright X^*_{unr}(M)\}/G \arrow[l, dashed, bend left, "p\text{-}\mathrm{Ind}"] & =: V_{\text{Bernstein}}/\mathbb{C} \\
\mathrm{Irr}_{temp}(G) \arrow[u, hook] \arrow[r, two heads, "ds"] & \{(M, \omega) \curvearrowright X^*_{unr, unitary}(M)\}/G \arrow[u, "sc"] \arrow[l, dashed, bend left, "p\text{-}\mathrm{Ind}"] & =: V_{HC}/\mathbb{R}
\end{tikzcd}$$

Principal block

$(T, triv)$, k nonarch.

$sc(St) = \{q^{-1/2}, q^{1/2}\}$

$ds(St) = St$



$G = SL_2$

$X^*_{unr}(T) \simeq \mathbb{C}^\times$

$\curvearrowleft \mathbb{Z}/2\mathbb{Z}$

$(z \longmapsto z^{-1})$

Planchérel formula (Harish-Chandra, '80s, Waldspurger '02)

$$\underset{\text{HC-Schwartz}}{\mathcal{S}(G)} \ni f \longmapsto \hat{f} \in \Gamma^{\infty}(p\text{-Ind}, \mathrm{Irr}_{temp}(G))$$

as in classical analysis

Fourier inversion:

(AKA "Plancherel formula")

$$f(x) = \int_{V_{HC}} \operatorname{tr} \hat{f}(\omega)(x) \underbrace{\deg(\omega)\, \mu_G(\omega)}_{\text{Plancherel measure}} \underset{\text{Haar measure}}{d\omega}$$

$\deg(\omega)$: <u>formal degree</u> (ω disc. ser.)

$\mu_G(\omega)$: <u>HC's μ-function</u>

Yu supercuspidals (k nonarch.)

$$\text{Yu}: \Phi = \left(\begin{array}{l} G = G_1 \supseteq G_2 \supsetneq \cdots \supsetneq G_{n+1}, \quad x \in \mathcal{B}(G_{n+1}), \\ \qquad\qquad \phi_1 \downarrow \quad \cdots \quad \phi_n \downarrow \qquad \kappa \in \mathrm{Irr}(G_{n+1,[x]}) \\ \qquad\qquad\quad \mathbb{C}^\times \qquad\quad\; \mathbb{C}^\times \end{array} \right) \rightsquigarrow \pi_\Phi \text{ supercuspidal}$$

Depth: $r_1 > \cdots > r_n > 0$

Kim, Fintzen: All supercuspidals arise this way if $p \nmid |\text{Weyl gp}|$.

Thm(s) ($G = G_{der}$)

$\exp_q(t) = q^t$

$$\deg(\pi_\Phi) = \frac{\dim \kappa}{[G_{n+1,x} : G_{n+1,x,0}]} \cdot \exp_q \tfrac{1}{2}\left(\dim G + \dim G_{n+1,x,0:0+} + \sum_{i=1}^{n} r_i \left(|R_{i+1}| - |R_{i+2}| \right) \right).$$

III. Local factors

and

the Plancherel measure

Some conjectures

$$\gamma(s,\varphi,r) := \varepsilon(s,\varphi,r)\cdot \frac{L(1-s,\varphi^\vee,r)}{L(s,\varphi,r)}.$$ (k local)

Conjecture: Say $\underset{\text{discrete series}}{\pi \in \mathrm{Irr}(M)}$ (π: V_{HC}; M: Levi of G) $\longleftrightarrow \begin{pmatrix} \varphi : WD_k \longrightarrow {}^L M, \\ \rho \in \mathrm{Irr}(S_\varphi) \end{pmatrix}$.

① $\mu_G(\pi) = |\gamma(0,\varphi,\hat{\mathfrak{g}}/\hat{\mathfrak{m}})|$. (Langlands)

② $\deg(\pi) = \dfrac{\dim(\rho)}{|S_\varphi^\natural|}|\gamma(0,\varphi,\hat{\mathfrak{m}}/\hat{\mathfrak{z}}^\Gamma)|$. (Hiraga, Ichino, Ikeda)

Regular/nonsingular supercuspidals

(k nonarch.)

Kaletha: LLC for Yu supercuspidals that are

① regular

$(\Downarrow)$

② nonsingular

Genericity condition in depth zero

L-packet has only supercuspidals

Thm: Kaletha's regular (S) and nonsingular (Ohara) L-packets satisfy the formal degree conjecture.

Future directions (k nonarch.)

Thm (Macdonald): Spherical Plancherel measure (G split):

$$\mu(\omega_s) \doteq \prod_{\alpha \in \Phi} \frac{(1-q^{-1-s(\alpha^\vee)})^{-1}}{(1-q^{-s(\alpha^\vee)})^{-1}}.$$

up to constant

Generalize to other blocks using Hecke algebra methods.