Variational and jump inequalities

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Lépingle's inequality

Theorem (Lépingle, 1976)

Let $f = (f_t)$ *be a martingale. For* $1 < p < \infty$ *and* $2 < r$ *we have*

$$\|V_t^r f_t\|_p \leq C_{p,r} \|f\|_p,$$

where V^r *is the* r-variation norm

$$\mathrm{V_t^r f_t} := \sup_{\mathrm{t(0)<\cdots<t(J)}} \Big(\sum_{\mathrm{j}} |\mathrm{f_{t(j+1)}} - \mathrm{f_{t(j)}}|^{\mathrm{r}} \Big)^{1/\mathrm{r}}.$$

- refines martingale maximal inequality: $Mf \leq f_0 + V_t^r f_t$
- quantifies martingale convergence: $V^r f_t$ finite $\implies f_t$ converges
- V^r is a parametrization-invariant version of $1/r$-Hölder continuity

Some variational estimates in harmonic analysis

Theorem (Jones+Seeger+Wright 2008)

If T_t are truncations of a cancellative singular integral, then

$$||V^r T_t f||_p \leq C_{p,r} ||f||_p, \quad 1 < p < \infty, r > 2.$$

Same for truncated Radon transforms along homogeneous curves.
Same for spherical averages on $\mathbb{R}^d$ for $\frac{d}{d-1} < p < 2d$.
They also prove an $r = 2$ "jump" endpoint to be explained in the next slide.

Theorem (Mas+Tolsa 2011, 2015)

Let μ be an n-dimensional AD regular Radon measure on $\mathbb{R}^d$. TFAE:

1. *μ is uniformly n-rectifiable*
2. *for any odd CZ kernel $V_t^r T_t$ is L^p bounded for $1 < p < \infty$, $r > 2$,*
3. *$V_t^r R_t$ is L^2 bounded for some $r < \infty$, where R_t are truncated Riesz transforms.*

Lépingle's inequality, endpoint version

Theorem (Pisier, Xu 1988/Bourgain 1989)

For $1 < p < \infty$ *we have the* jump inequality

$$J_2^p(f_t) := \sup_{\lambda>0} ||\lambda N_\lambda^{1/2} f_t||_p \leq C_p ||f||_p,$$

where N_λ *is the* λ-jump counting function

$$N_\lambda f_t := \sup_{t(0)<\cdots<t(J)} \#\{j \mid |f_{t(j+1)} - f_{t(j)}| > \lambda\}.$$

Observation

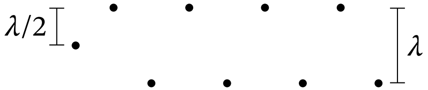
$$||V^r f_t||_{p,\infty} \leq C_{p,r} \sup_{\lambda>0} ||\lambda N_\lambda^{1/2} f_t||_{p,\infty}, \quad 2 < r.$$

This + real interpolation shows that jump inequalities imply r-variational estimates in open ranges of p.

Proof of endpoint Lépingle inequality

λ-jump counting function is morally extremized by
greedy selection of $\lambda/2$-jumps:

$$t(0) := 0, \quad t(j+1) := min\{s > t(j) \mid |f_s - f_{t(j)}| > \lambda/2\}.$$



$$\lambda N_\lambda^{1/2} \le \lambda\Big(\sum_j \frac{|f_{t(j+1)} - f_{t(j)}|^2}{(\lambda/2)^2}\Big)^{1/2} \le 2\Big(\sum_j |f_{t(j+1)} - f_{t(j)}|^2\Big)^{1/2}$$

– square function of the stopped martingale $f_{t(j)}$, bounded on L^p.

Remark (vector valued)

For martingales with values in a Banach space with martingale cotype q can have power $1/q$ instead of $1/2$.

Jumps as a real interpolation space

Proof of Lépingle's inequality gives for a given λ a decomposition

$$f_t = \sum_j \mathbf{1}_{t(j)\le t<t(j+1)} f_{t(j)} + \sum_j \mathbf{1}_{t(j)\le t<t(j+1)} (f_t - f_{t(j)}).$$

Observation (Pisier+Xu 1988)

This decomposition shows in fact that

$$[L^\infty(V^\infty), L^1(V^1)]_{1/2,\infty}(f_t) \lesssim \|f\|_2,$$

where the LHS is a norm in a real interpolation space.

More generally, it turns out that

$$J_2^p(f_t) \sim [L^\infty(V^\infty), L^{p\theta}(V^{2\theta})]_{\theta,\infty}(f_t) \lesssim \|f\|_p$$

for $1 < p < \infty$ and $0 < \theta < 1$.

Application: diffusion semigroups

Corollary

If (T_t) is a diffusion semigroup (i.e., contractive on L^1 and L^∞, selft-adjoint, order positive, $T_t\mathbf{1} = \mathbf{1}$), then

$$J_2^p(T_t f) \leq C_p \|f\|_p, \quad 1 < p < \infty.$$

Proof.

Rota's dilation theorem: $T_t f = \mathbb{E} \circ$ martingale.
Conditional expectation bounded on J_2^p by interpolation. □

Corollary (Mirek, Stein, ZK)

Let $G \subset \mathbb{R}^d$ be a symmetric convex body and $A_t f(x) = |G|^{-1} \int_G f(x + ty) dy$. Then

$$J_2^p(A_t f) \leq C_p \|f\|_p, \quad 3/2 < p < 4.$$

- maximal estimate by Bourgain (L^2), Carbery

Periodic multipliers

Let (m_t) be a sequence of multipliers supported on $[-\frac{1}{2q}, \frac{1}{2q}]^d$, q positive integer. Define periodic multipliers

$$m_t^{per}(\xi) := \sum_{l \in \mathbb{Z}^d} m_t(\xi - l/d).$$

Theorem (Magyar+Stein+Wainger 2002)

For any Banach space X of functions in t and $1 \leq p \leq \infty$ we have

$$\|m^{per}\|^{mult}_{\ell^p \to \ell^p(X)} \leq C_{p,d} \|m\|^{mult}_{L^p \to L^p(X)}$$

Theorem (Mirek+Stein+ZK)

For any Banach spaces X_0, X_1 of functions in t and $1 \leq p\theta$ we have

$$\|m^{per}\|^{mult}_{\ell^p \to [\ell^\infty(X_0), \ell^{p\theta}(X_1)]_{\theta;\infty}} \leq C_{p,d} \|m\|^{mult}_{L^p \to [L^\infty(X_0), L^{p\theta}(X_1)]_{\theta;\infty}}$$

Corollary

Application: discrete Radon transforms

Let $A_N f(x) := \frac{1}{N} \sum_{n=1}^{N} f(x - n^2)$.

Theorem (Mirek+Stein+Trojan 2015)

$$\|V_N^r A_N f\|_{\ell^p(\mathbb{Z})} \lesssim \|f\|_{\ell^p(\mathbb{Z})}, \quad 1 < p < \infty, \quad r > 2.$$

- Circle method approach by Bourgain
- Ionescu–Wainger multipliers select rationals with small denominators
- Use periodic multipliers on major arcs

Theorem (Mirek+Stein+ZK)

$$J_2^p(A_N f) \lesssim \|f\|_{\ell^p(\mathbb{Z})}, \quad 1 < p < \infty.$$

What are correct endpoint variational inequalities?

Theorem (S.J. Taylor 1972)

If (B_t) is the standard Brownian motion, then

$$\psi(V_{t<T})(B_t) = \sup_{t_0<\cdots<t_J<T} \|B_{t_{j+1}} - B_{t_j}\|_{\psi(L)_j},$$

is a.s. finite with the Young function

$$\psi(t) = t^2 / \log_* \log_* t.$$

Same is true for all martingales with continuous paths, since they are reparametrizations of Brownian motion.

Question

What is the best ψ-variational estimate for general martingales?

Variational inequalities:

$$\psi(t) = t^r, r > 2.$$

Jump inequalities:

$$\psi(t) = t^2/(\log_* t)^{1+\epsilon}.$$

Variational estimates in time-frequency analysis

Theorem (Oberlin+Seeger+Tao+Thiele+Wright 2009)

The variationally truncated partial Fourier integral

$$\sup_{t_0<\cdots<t_J}\Big(\sum_j\Big|\int_{t_j<\xi<t_{j+1}} e^{2\pi i x\xi}\hat{f}(\xi)d\xi\Big|^r\Big)^{1/r}$$

is bounded $L^2 \to L^2$ for $r > 2$.

- Quantitative form of Carleson's theorem

Theorem (Do+Muscalu+Thiele 2016)

The variationally truncated bilinear Hilbert transform

$$\sup_{t_0<\cdots<t_J}\Big(\sum_j\Big|\int_{t_j<\xi_1<\xi_2<t_{j+1}} e^{2\pi i x(\xi_1+\xi_2)}\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)d\xi_1 d\xi_2\Big|^{r/2}\Big)^{2/r}$$

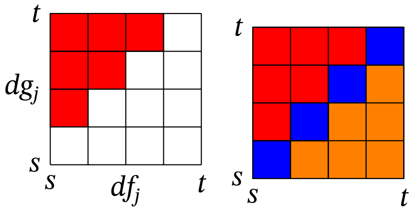
is bounded $L^2 \times L^2 \to L^1$ for $r > 2$.

- Uses a variational estimate for paraproducts

Martingale paraproduct

For martingales $(f_j)_j$, $(g_j)_j$ and martingale differences $df_j = (f_j - f_{j-1})$ the *truncated paraproduct* (or *area process*) is defined by

$$\Pi_s^t(f,g) := \sum_{s \le j < k \le t} df_j dg_k.$$



$$(f_t - f_s)(g_t - g_s) = \color{red}{\Pi_s^t(f,g)} + \color{blue}{df_{s+1}dg_{s+1} + \cdots + df_t dg_t} + \color{orange}{\Pi_s^t(g,f)}$$

Variational estimate for martingale paraproduct

Theorem (Do+Muscalu+Thiele 2012 (doubling), Kovač+ZK 2018 (non-doubling))

For $1 < p_1, p_2 < \infty$ *with* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ *and* $2 < r$ *we have*

$$\Big\| \sup_{t_0 < \cdots < t_J} \Big(\sum_j \big| \Pi_{t(j)}^{t(j+1)}(f,g) \big|^{r/2} \Big)^{2/r} \Big\|_{p_3'} \leq C_{p_1,p_2} \|f\|_{p_1} \|g\|_{p_2}$$

Proof idea: for $\lambda > 0$ estimate the jump counting function

$$\sup_{t(0) < \cdots < t(J)} \#\{ j \mid |\Pi_{t(j)}^{t(j+1)}(f,g)| > \lambda \}.$$

Application: stochastic integrals

Corollary

Let (X_t), (Y_t) be càdlàg continuous time martingales. Then for $1 < p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $2 < r$ we have

$$\Big\| \sup_{t_0 < \cdots < t_J} \Big(\sum_j \Big| \int_{(t(j), t(j+1)]} (X_{s-} - X_{t(j)}) dY_s \Big|^{r/2} \Big)^{2/r} \Big\|_{p_3'} \leq C_{p_1, p_2, r} \|X\|_{p_1} \|Y\|_{p_2}.$$

- Chevyrev+Friz 2018: diagonal case $p_1 = p_2$.
- Friz+Victoir 2006: martingales with continuous paths.
- Classically X, Y are Brownian motions.
- Useful in Lyons's theory of rough paths.