An introduction to equivariant homotopy theory

Groups

Consider compact Lie groups G
and their closed subgroups H.

Examples:

1. $\mathbb{Z}/2$
2. finite groups
3. S^1

G-spaces

spaces with a continuous left action
(if pointed, basepoint fixed by G)

G-CW complexes

$G/H \times D^n, G/H \times S^{n-1}$

$G/H_+ \wedge D^n, G/H_+ \wedge S^{n-1}$

Equivariant homotopy
study G-maps up to G-homotopy, $[X, Y]_G$

Consider
$[G/H_+ \wedge S^n, X]_G \cong [S^n, X]_H \cong [S^n, X^H]$

X^H here is the fixed point space, $X^H \cong \mathrm{map}_G(G/H, X)$

Definition. A G-map $f : X \to Y$ is a *weak G-equivalence* if $f^H : X^H \to Y^H$ is a weak equivalence for all closed subgroups H.

That is, f_* is a weak G-equivalence if it induces an isomorphism on $\pi_n^H(-) = [G/H_+ \wedge S^n, -]_G$ for all H and n.

G-Whitehead Theorem. A weak G-equivalence of G-CW complexes is a G-homotopy equivalence.

Diagrams over the orbit category

Definitions.
1. The *orbit category*, written $\mathcal{O}_G$, is a topological category with objects $\{G/H\}$ and morphisms the spaces of G-maps:
$\mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K) \cong \mathrm{map}_G(G/H, G/K) \cong (G/K)^H$

2. An $\mathcal{O}_G$*-space* is a contravariant functor from $\mathcal{O}_G$ to spaces.

3. Any G-space X has an associated $\mathcal{O}_G$-space, ΦX with:

$$\Phi(X)(G/H) = X^H$$

Since $X^H = \mathrm{map}_G(G/H, X)$, functoriality follows by composition.

Theorem. The functor $\Phi(-) = \mathrm{map}_G(G/H, -)$ induces an equivalence between the homotopy theories of G-spaces and $\mathcal{O}_G$-spaces.

Examples.

1. Let $\mathcal{F}$ be a *family* of subgroups (closed under conjugation and subgroups). Define an $\mathcal{O}_G$ space by:

$$(E\mathcal{F})^H = \begin{cases} \mathrm{pt} & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}$$

Note $E\{1\} \simeq EG$, the universal free G-space.

2. Define a *coefficient system* to be a functor $\underline{\pi} : \mathcal{O}_G \to h\mathcal{O}_G \to \mathrm{Ab}$. There is an associated *Eilenberg-Mac Lane* $\mathcal{O}_G$-space:

$$K(\underline{\pi}, n)(G/H) = K(\underline{\pi}(G/H), n)$$

There is an associated "ordinary" equivariant cohomology theory on G-spaces:

$$\widetilde{H}^n(X; \underline{\pi}) = [X, K(\underline{\pi}, n)]_G.$$

G-equivariant spectra

Definitions. A *G-universe* $\mathcal{U}$ is a countably infinite dimensional real inner product space with an action of G through linear isometries. Let V, W be finite dimensional sub G-inner product spaces of $\mathcal{U}$.

Define $W - V$ to be the orthogonal complement of V in W.

Let S^V be the one-point compactification of V.

A *"G-equivariant spectrum"* X is a collection of G-spaces $X(V)$ for each V in $\mathcal{U}$ with structure maps $\Sigma^{W-V}X(V) \to X(W)$ (or their adjoints, $X(V) \to \Omega^{W-V}X(W)$.)

Example. $\Sigma^\infty X$, the *suspension spectrum* of a pointed G-space X, has $\Sigma^\infty X(V) = S^V \wedge X$.

Stable orbit category

Definitions.

1. The *stable orbit category* $\mathcal{O}S_G$ is a spectral category; it is the full subcategory of G-spectra with objects $\Sigma^\infty G/H_+$ and morphisms the spectrum of G-maps:

$$F_G(\Sigma^\infty G/H_+, \Sigma^\infty G/K_+) \cong (\Sigma^\infty G/K_+)^H$$

2. An $\mathcal{O}S_G$*-module* is a contravariant functor from $\mathcal{O}S_G$ to spectra.

3. Any G-spectrum X has an associated $\mathcal{O}S_G$-module, ΦX with:

$$\Phi(X)(G/H) = X^H$$

Since $X^H = F_G(\Sigma^\infty G/H_+, X)$, functoriality follows by composition.

Theorem. (tom Dieck) For based G-CW complexes X, there is a natural equivalence

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} X^H),$$

where $WH = NH/H$ and $Ad(WH)$ is its adjoint representation; the sum runs over all conjugacy classes of subgroups H in G.

For example

$$F_G(\Sigma^\infty G/G_+, \Sigma^\infty G/e_+) \cong (\Sigma^\infty G/e_+)^G \not\simeq \emptyset$$

Theorem. The functor $\Phi(-) = F_G(\Sigma^\infty G/H_+, -)$ induces an equivalence between the homotopy theories of G-spectra and $\mathcal{O}S_G$-modules.

Example. Define a *Mackey functor* to be a functor $M : \mathcal{O}S_G \to h\mathcal{O}S_G \to \mathrm{Ab}$. There is an associated *Eilenberg-Mac Lane* $\mathcal{O}S_G$-module:

$$HM(G/H) = K(M(G/H), 0)$$

There is an associated equivariant cohomology theory:

$$\widetilde{H}^n(X; M) = [X, \Sigma^n HM]_G.$$

This cohomology theory is defined on all G-spectra and is in fact $\mathrm{RO}(G)$-graded. That is, for any real representation V,

$$\widetilde{H}^V(X; M) = [X, \Sigma^V HM]_G.$$

Rational Equivariant Spectra

Theorem. (Greenlees-May '95) Let G be finite. Then, for any rational G-spectrum X, there is a natural equivalence

$$X \xrightarrow{\simeq} \prod \Sigma^n H(\underline{\pi}_n X).$$

Theorem. Let G be finite. The homotopy theory of *rational G-spectra* is modeled by *differential graded rational Mackey functors.* Moreover, the derived category is equivalent to the graded category.

Proof.

$$\begin{aligned} \mathbb{Q}\text{ - }G\text{ -spectra} &\simeq_Q \text{Mod-}(H\mathbb{Q} \wedge \mathcal{O}S_G) \\ &\simeq_Q d.g.\,\text{Mod-}\,\Theta(H\mathbb{Q} \wedge \mathcal{O}S_G) \\ &\simeq_Q d.g.\,\text{Mod-}\,h\mathcal{O}S_G. \end{aligned}$$

Since $\pi_* \operatorname{Hom}(\Sigma^\infty G/H, \Sigma^\infty G/K) \otimes \mathbb{Q} = 0$ for $* \neq 0$, then $H_0\Theta(H\mathbb{Q} \wedge \mathcal{O}S_G) \cong h\mathcal{O}S_G$.

The last statement follows since rational Mackey functors are all projective and injective.

Theorem. Let G be a compact Lie group. The homotopy theory of rational G-spectra is modeled by differential graded modules over a rational DGA whose homology is isomorphic to the homotopy of the rational stable orbit category $\mathcal{O}S_G$.

Proof.

$$\begin{aligned} \mathbb{Q}\text{ - }G\text{ -spectra} &\simeq_Q \text{Mod-}(H\mathbb{Q} \wedge \mathcal{O}S_G) \\ &\simeq_Q d.g.\,\text{Mod-}\,\Theta(H\mathbb{Q} \wedge \mathcal{O}S_G) \end{aligned}$$

Here $\pi_* \mathcal{O}S_G \otimes \mathbb{Q} \cong H_* \Theta(H\mathbb{Q} \wedge \mathcal{O}S_G)$.

Note.
An algebraic model, but not useful for calculations.

Conjecture.(Greenlees) For any compact Lie group G there is an abelian category $\mathcal{A}(G)$ such that

$$\mathbb{Q}\text{ - }G\text{-spectra} \simeq \text{d. g. } (\mathcal{A}(G))$$

where $\mathcal{A}(G)$ has injective dimension equal to the rank of G.