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Why $H\mathbb{Z}$-algebra spectra are differential graded algebras ?

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Motivation and Context

In homological algebra, to understand a commutative ring R, one studies the category of its modules, R-Mod, the associated chain complexes or monoids, the differential graded R-algebras. Since the structure of R-Mod is often too rigid to efficiently work in, it becomes more appropriate to operate in the derived category $\mathcal{D}(R)$, which is the homotopy category of differential graded R-modules, $\mathrm{Ho}(DGR\text{-}Mod)$.

A generalization of homological algebra is offered by algebra of symmetric spectra. In this frame, symmetric spectra take the place of abelian groups, and the analogy in terms of symmetric monoidal structures goes as follows:

Homological algebra	**Algebra of spectra**
$(Ab, \otimes, \mathbb{Z})$	$(Sp^{\Sigma}, \wedge, \mathbb{S})$
R, ring	HR, Eilenberg - Mac Lane ring spectrum.

A well-behaved symmetric smash product $\wedge$ on spectra took time to be elaborated, but, in return, it made possible clear categorical definitions of ring, module and algebra spectra. Before, the notions of spectra and ring spectra already existed, but all algebraic structures had complex, up to homotopy, properties.

In the older context, Alan Robinson [Rob] established a connection between rings and ring spectra. He defined a notion of A_∞-modules over the ring spectrum HR, and showed that, up to a suitable notion of homotopy, the category of A_∞-modules over HR is equivalent to $\mathcal{D}(R)$. Nevertheless, it was difficult to obtain a similar result for algebras because of involved definitions. However, this became achievable in the modern setting of algebra of spectra, where the homotopy theory is encoded in Quillen model structures.

After strengthening the result of Robinson by showing that the category of HR-module spectra is Quillen equivalent to DGR-Mod

$$\mathcal{D}(R) = \mathrm{Ho}(DGR\text{-}Mod) \qquad \sim \qquad \mathrm{Ho}(HR\text{-module spectra}),$$

Brooke Shipley [Shi, Theorem 1.1] showed that the HR-algebra spectra capture the same "up to homotopy" information as differential graded R-algebras:

Theorem 1 (Shipley) *For any discrete commutative ring R, the model categories of unbounded differential graded R-algebras and HR-algebra spectra are Quillen equivalent. The associated composite derived functors are denoted*

$$\mathcal{H} : DG\text{-}Alg_R \to HR\text{-}Alg_{Sp^\Sigma} \text{ and } \Theta : HR\text{-}Alg_{Sp^\Sigma} \to DG\text{-}Alg_R.$$

In other words, this theorem states an equivalence of categories

$$\mathrm{Ho}(DG\text{-}Alg_R) \qquad \sim \qquad \mathrm{Ho}(HR\text{-algebra spectra}).$$

The aim of this Master thesis was to acquire a sufficient knowledge of algebra of spectra and of model category theory in order to understand the above result, and to explain the essential arguments employed in the proof. For simplicity, we concentrated on the case $R = \mathbb{Z}$.

A Short Useful Reminder

Definition 1 *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and K an object in $\mathcal{C}$. The category of* ***symmetric spectra over $\mathcal{C}$ with respect to*** *K, denoted $Sp^\Sigma(\mathcal{C}, K)$, is the category of modules over the commutative monoid $Sym(K) := (I, K, K \otimes K, ..., K^{\otimes n}, ...)$ in the category $(\mathcal{C}^\Sigma, \otimes, I)$ of symmetric sequences on $\mathcal{C}$.*

If $\mathcal{C}$ is closed symmetric monoidal, then so is the category $(Sp^\Sigma(\mathcal{C}, K), \wedge, \mathrm{Sym}(K))$. For two spectra X and Y, their symmetric **smash product** $X \wedge Y$ is given by the coequalizer $\mathrm{coeq}(X \otimes \mathrm{Sym}(K) \otimes Y \rightrightarrows X \otimes Y)$.

This smash product allows one to define a **symmetric ring spectrum** R to be simply a monoid in $(Sp^\Sigma(\mathcal{C}, K), \wedge, \mathrm{Sym}(K))$. Similarly, an **$R$-algebra spectrum** is a monoid in $(R\text{-}Mod_{Sp^\Sigma(\mathcal{C},K)}, \wedge_R, R)$, for a symmetric ring spectrum R.

Definition 2 *The* ***Eilenberg - Mac Lane spectrum*** *$H\mathbb{Z}$ over pointed simplicial sets is defined at level n by $H\mathbb{Z}_n = U\tilde{\mathbb{Z}}S^n$. It is the underlying simplicial set of the reduced free simplicial abelian group generated by the n-sphere S^n.*

Definition 3 *An adjoint pair of functors (F, G) between two model categories $(\mathcal{C}, WE_\mathcal{C}, Fib_\mathcal{C}, Cof_\mathcal{C})$ and $(\mathcal{D}, WE_\mathcal{D}, Fib_\mathcal{D}, Cof_\mathcal{D})$ is called a* ***Quillen equivalence****, if $F(Cof_\mathcal{C}) \subseteq Cof_\mathcal{D}$, $G(Fib_\mathcal{D}) \subseteq Fib_\mathcal{C}$ and if furthermore for all cofibrant A in $\mathcal{C}$ and fibrant B in $\mathcal{D}$*

$$f : FA \to B \in WE_\mathcal{D} \iff f^\sharp : A \to GB \in WE_\mathcal{C},$$

where $f^\sharp$ is the adjoint of f.

Finally, the context requires the categories involved to combine both the monoidal and model structure in an appropriate way. Since the categories all have cofibrant units here, the compatibility condition comes down to the following.

Definition 4 *A model category $(\mathcal{C}, WE_\mathcal{C}, Fib_\mathcal{C}, Cof_\mathcal{C})$ is* ***model monoidal*** *if it is endowed with a closed symmetric monoidal structure $(\mathcal{C}, \otimes, I)$ and satisfies the* ***push-out product axiom*** *:*
Let $f : A \rightarrowtail B$ and $g : K \rightarrowtail L$ be cofibrations in $\mathcal{C}$. Then the map $f \square g : A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$ induced by the pushout diagram

$$\begin{tikzcd}
A \otimes K \arrow[r, "f \otimes K"] \arrow[d, "A \otimes g"'] & B \otimes K \arrow[d, "j_0"] \arrow[ddr, bend left, "B \otimes g"] & \\
A \otimes L \arrow[r, "j_1"] \arrow[drr, bend right, "f \otimes L"'] & A \otimes L \coprod_{A \otimes K} B \otimes K \arrow[dr, dashed, "f \square g"] & \\
& & B \otimes L
\end{tikzcd}$$

is also a cofibration. If in addition one of the maps f or g is a weak equivalence, then so is the map $f \square g$.

Strategy of the Proof

The strategy Shipley uses to prove the Quillen equivalence in Theorem 1 is the following :

1. study first the relation between $H\mathbb{Z}$-module spectra and differential graded modules;
2. use then the fact that $H\mathbb{Z}\text{-}Alg_{Sp^\Sigma}$ and $DG\text{-}Alg_\mathbb{Z}$ are the categories of *monoids* in $H\mathbb{Z}\text{-}Mod_{Sp^\Sigma}$ and $DG\mathbb{Z}$-Mod respectively.

To make a connection between $H\mathbb{Z}\text{-}Mod_{Sp^\Sigma}$ and $DG\mathbb{Z}$-Mod, we will need to consider two intermediate categories, the category $Sp^\Sigma(sAb)$ of symmetric spectra over simplicial abelian groups and the category $Sp^\Sigma(Ch_+)$ of symmetric spectra over non-negative chain complexes.

We start with a chain of three more or less "classical" adjoint pairs, defined on the underlying categories

$$sSet_* \underset{U}{\overset{\tilde{\mathbb{Z}}}{\rightleftarrows}} sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch_+ \underset{C_0}{\overset{i}{\rightleftarrows}} DG\mathbb{Z}\text{-}Mod.$$

Lifting these adjunctions to the categories of corresponding symmetric sequences, and then to spectra, gives the following zig-zag of adjoint pairs

$$H\mathbb{Z}\text{-}Mod_{Sp^\Sigma} \underset{U}{\overset{Z}{\rightleftarrows}} Sp^\Sigma(sAb) \underset{\varphi^*N}{\overset{L}{\leftrightarrows}} Sp^\Sigma(Ch_+) \underset{R}{\overset{D}{\rightleftarrows}} DG\mathbb{Z}\text{-}Mod,$$

which lies at the heart of the proof.
For this zig-zag to be coherent, two types of conditions must be satisfied.

- **Conditions on the categories:** they summarize in demanding compatibility between the model and the monoidal structure;
- **Conditions on the adjoint functor pairs:** they require the monoidal-model category structures be appropriately transferred via the zig-zag.

The desired Quillen equivalence will follow from the fact that the four categories and their adjoint pairs in the main zig-zag satisfy key Theorems 2 and 3 opposite. This will imply that the corresponding categories of monoids – the $H\mathbb{Z}$-algebra spectra and the differential graded algebras – are equipped with induced model structures.

The three adjoint pairs (Z, U), (L, φ^*N), (D, R) from the zig-zag induce Quillen equivalences on the categories of monoids as follows:

$$H\mathbb{Z}\text{-}Alg_{Sp^\Sigma} \underset{U}{\overset{Z}{\rightleftarrows}} Mon_{Sp^\Sigma(sAb)} \underset{\varphi^*N}{\overset{L^{mon}}{\leftrightarrows}} Mon_{Sp^\Sigma(Ch_+^\Sigma)} \underset{R}{\overset{D}{\rightleftarrows}} DG\text{-}Alg_\mathbb{Z},$$

and the desired functors $\mathcal{H}$: $DG\text{-}Alg_R \to HR\text{-}Alg_{Sp^\Sigma}$ and Θ: $HR\text{-}Alg_{Sp^\Sigma} \to DG\text{-}Alg_R$ emerge as the composite total derived functors induced on the correspondent homotopy categories.

Key Theorems

In [SS00, Theorem 4.1], Schwede and Shipley establish what are the necessary conditions on a category $\mathcal{C}$ to define the induced model structures on the associated categories of monoids, R-modules and R-algebras. We mention here only the part concerning algebras, where taking $R = I$ will give the result for $Mon_\mathcal{C}$.

Theorem 2 (Schwede and Shipley) *Let $\mathcal{C}$ be a cofibrantly generated, monoidal model category, satisfying the monoid axiom and such that every object in $\mathcal{C}$ is small. Let R be a commutative monoid in $\mathcal{C}$. Then the category $R\text{-}Alg_\mathcal{C}$ of R-algebras over $\mathcal{C}$ is a cofibrantly generated model category.*

Sufficient conditions for extending Quillen equivalences between two monoidal model categories on the associated categories of monoids are given by Schwede and Shipley in [SS03, Theorem 3.12]:

Theorem 3 (Schwede and Shipley) *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal model categories and $R : \mathcal{C} \to \mathcal{D}$ be the right adjoint of a weak monoidal Quillen pair. Suppose that the unit objects in $\mathcal{C}$ and $\mathcal{D}$ are cofibrant. If the forgetful functor creates model structures for monoids in $\mathcal{C}$ and $\mathcal{D}$, then the adjoint pair*

$$L^{mon} : Mon_\mathcal{D} \rightleftarrows Mon_\mathcal{C} : R^{mon}$$

is a Quillen equivalence.

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