"THE HOMOTOPY TYPE OF THE COBORDISM CATEGORY" PROPOSAL FOR A SEMINAR PROGRAM WINTERSEMESTER 2007/08

The main aim of the seminar will be to understand the main theorem in [GMTW07] and its proof.

Theorem 0.1. *There is a weak equivalence*

$$B\mathcal{C}_d \simeq \Omega^{\infty-1} MT(d).$$

Proof. There is the following chain of weak equivalences

$$
\begin{aligned}
B\mathcal{C}_d &\simeq B|C^\infty(-,\mathcal{C}_d)| && (0.2)\\
&\cong B|C_d(-)| && (0.3)\\
&\xrightarrow{\simeq} B|C_d^\pitchfork(-)| && (0.4)\\
&\xleftarrow{\simeq} B|D_d^\pitchfork(-)| && (0.5)\\
&\simeq |\beta D_d^\pitchfork(-)| && (0.6)\\
&\xrightarrow{\simeq} |D_d(-)| && (0.7)\\
&\xrightarrow{\simeq} \Omega^{\infty-1} MT(d). && (0.8)
\end{aligned}
$$

□

This is an admittedly very technocratic overview of the proof. The purpose of our seminar is to fill these symbols with life. Of course details can be found in [GMTW07] but here are some first explanations.
About the objects used above:

- $\mathcal{C}_d$ is the category of embedded cobordisms, a topological category.
- $C^\infty(-,\mathcal{C}_d)$, $C_d(-)$, $C_d^\pitchfork(-)$ and $D_d^\pitchfork(-)$ are CAT-valued sheaves, i.e. sheaves on the category $\mathcal{X}$ of smooth manifolds without boundary which take values in the category of (small and discrete) categories.
- $\beta D_d^\pitchfork(-)$ and $D_d(-)$ are SET-valued sheaves.
- $\Omega^{\infty-1} MT(d)$ is the infinite loop space associated to a suitable Thom-spectrum.

One uses the following constructions:

- Given a SET-valued sheaf $\mathcal{F}(-)$ one can form the topological space
  $$|\mathcal{F}(-)| = |[n] \mapsto \mathcal{F}(\Delta_n^e)|,$$
  i.e. the geometric realization of the simplicial set obtained by evaluating on the extended standard simplices.
- Given a CAT-valued sheaf $\mathcal{F}$ one can similarly form $|\mathcal{F}(-)|$, a topological category.
- Given a topological category $\mathcal{C}$ one can form its classifying space
  $$B\mathcal{C} = |N_\bullet \mathcal{C}|,$$
  a topological space.
- Given a CAT-valued sheaf $\mathcal{F}(-)$ the β-construction yields an associated SET-valued sheaf $\beta\mathcal{F}(-)$.

Date: October 19, 2007.

Talk 1 and 2: A geometric interpretation of maps into $\Omega^{\infty-1}MT(d)$

The equivalence (0.8): Introduce $\Omega^{\infty-1}MT(d)$, $D_d[X]$ and show

$$[X, \Omega^{\infty-1}MT(d)] \xrightarrow{\cong} D_d[X].$$

Compare [GMTW07, Theorem 3.4]. This is a parametrized version of the Pontryagin-Thom construction and uses Phillips' submersion theorem [Phi67]. Prove the natural bijection $[X, |\mathcal{F}(-)|] \cong \mathcal{F}[X]$ for every SET-valued sheaf $\mathcal{F}(-)$, where $\mathcal{F}[X]$ denotes concordance classes. Compare [MW04, Proposition A.1.1] and explain how (0.8) follows.

Talk 3 and 4: The cobordism category and its classifying space

Introduce the category of embedded cobordisms and explain (0.2) and (0.3), see [GMTW07, Theorem 2.9]. This includes presenting a variety of general things about nerves and classifying spaces on the abstract side and a careful discussion of the CAT-valued sheaf $C_d(-)$ on the geometric side.

Talk 5 and 6: Sheaf-theoretic tools and the remaining equivalences

Explain the relative surjectivity criterium [MW04, Proposition 2.4.4] and prove the equivalences (0.4) and (0.5). Explain the β-construction, which turns a CAT-valued sheaf $\mathcal{F}(-)$ into a SET-valued sheaf, such that $B|\mathcal{F}(-)| \simeq |\beta\mathcal{F}(-)|$. This explains (0.6). Discuss the remaining equivalence (0.7).

Talk 7 and 8: Group completion, the Mumford conjecture, graph cobordism

These talks should have a survey character. Survey group completion. Sketch how the Mumford conjecture is deduced in Section 6 and 7 of [GMTW07]. (This involves a new version of the equivalence (0.7).) The second talk could report on [Gal07], where a graph cobordism category is introduced and used to prove $\mathbb{Z} \times B\,\mathrm{Aut}_\infty^+ \simeq QS^0$. This is the analogue of the Mumford conjecture for (stabilized) automorphism groups of free groups instead of mapping class groups.

Every pair of talks above should be covered in one double-session. The fine-tuning between the two talks within one-session is left to the speakers. As a general fact there is never enough time to give complete proofs. “Report-style” should be prefered to “going over time”.

If you are interested in giving a talk simply send me an e-mail.

In hope for enthusiastic volunteers,

Holger Reich

References

[Gal07] S. Galatius. Stable homology of automorphism groups of free groups. arXiv:math/0610216v2, 2007.

[GMTW07] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss. The homotopy type of the cobordism category. arXiv:math/0605249v2, 2007.

[MW04] I. Madsen and M. Weiss. The stable moduli space of riemann surfaces: Mumford's conjecture. arXiv:math/0212321v3, 2004.

[Phi67] Anthony Phillips. Submersions of open manifolds. *Topology*, 6:171–206, 1967.

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