Research Seminar of the GRK 1150 Sommersemster 08
Homotopy and Cohomology

Algebraic K-theory of number fields (after A. Borel)

We fix a number field F and its ring of integers O_F. We assume that everybody is familiar with the definition of algebraic K-theory of rings using the *plus construction* and with the fact that the latter does not change the homology (cf the lectures of Holger Reich if you don't know this yet).

The goal of the seminar is to understand as much as possible of Borel's proof of following theorem [Bo2, Proposition 12.2]:

Theorem. *The ranks of the finite dimensional vector spaces $K_n(O_F) \otimes \mathbf{Q}$ are periodic of order 4 for $i \geq 2$ and given by $\dim K_n(O_F) = 0, r_1 + r_2, 0, r_2$ depending wether $i = 0,\ 1,\ 2,\ 3$ mod 4; where r_1 (resp. r_2) is the number of real (resp. pairs of conjugated complex) embeddings of F.*

1. Reduction to the real cohomology of $SL(F)$. Briefly discuss K_0 and K_1 (Dirichlet's units theorem, [Ne, I.7]). State the main theorem of [Qu3] and the isomorphism $K_*(O_F) \otimes \mathbf{Q} \cong K_*(F) \otimes \mathbf{Q}$ for $* \geq 2$ which follows from [Qu1] and [Qu2, Corollary p. 113]. Then relate the latter to the primitive elements of $H_*(GL(O_F), \mathbf{Q})$ using Milnor-Moore, further to the indecomposable elements of $H^*(GL(O_F), \mathbf{Q})$ and finally to $H^*(SL(O_F), \mathbf{R})$.

2. Square integrable forms. Explain the first two sections of [Bo2], in particular Proposition 2.5 of loc. cit.. For the Star-operator and some other definitions see e.g. [Wa].

3. The morphism j Part I. Introduce the morphism j as in section 3 of [Bo2, p.241/242]. In particular, explain the first and the second sentence in that section, as well as the fifth (this includes the definition of relative Lie algebra cohomology). See e. g. [Bo5, sections 1.6-1.8] and [BW, section 1] for some details. We do not need general E-coefficients.

4. The morphism j Part II. Explain as much as possible Matsushima's theorem and its variants as in [Bo2, p.243/244]. Additional information can be found in the original work of Matsushima or in [Bo5, section 2].

5. The morphism j Part III. Say as much as possible about the proof of Proposition 3.6 of loc. cit., including the results of the article of Garland which are used by Borel [Bo2, p.244/245].

6. Linear algebraic groups. Explain the fundamental definitions and results about algebraic groups with examples (including parabolic and Levi subgroups and root systems, etc). Briefly discuss the (easier and in many ways similar) analogues for Lie groups. Check what will be needed in talks 7 and 10. A possible reference is [Bo0].

7. Parabolic subgroups and Siegel sets. Provide a survey of [Bo2, sections 4 and 5]. Explain "parabolic" and "Levi"-subgroup and provide an example. Corollary 5.7 will be used in talk 8.

8. The Borel-Serre compactification. [Bo2, section 6] and the original article [BS]. Discuss examples, at least SL_2 (see the introduction of [BS]).

9. A square integrability criterion and the injectivity of j. The main result we will need later is [Bo2, Theorem 7.5], which follows from [Bo2, Proposition 3.6 and Theorem 7.4].

10. The constants $c(G)$ and $m(G(R))$. Prove the very last equality of [Bo2, section 9]. When talking about root systems etc, keep in mind that the only example we care about later is SL_n.

11. Bott periodicity and the final results. Discuss parts of Bott periodicity [B]. What we really need is the rational cohomology rings of SU and SU/SO. Explain the isomorphisms [Bo2, 10.2.(3),10.6 (1)] for $H_n = SL_n(\mathbf{R}), SL_n(\mathbf{C})$, see [Ca]. Then use the previous results (in particular [Bo2, 7.5, 9.5, 10.6] to prove [Bo2, Theorem 11.1] and - finally - the desired computation of $H^*(SL(F), \mathbf{R})$ [Bo2][11.5(3)].

12. The Borel regulator. In this survey talk, the Borel regulator and some of its (conjectural) properties should be introduced (see [Bo4], [Bu]).

The first speaker will show how to reduce the problem of computing the groups $K_n(F) \otimes \mathbf{Q}$ to the computation of the cohomology rings $H^*(SL(O_F), \mathbf{R})$.

The next talks 2-11 are devoted to this computation. The introduction of [Bo1] is very clear, and gives a good idea of the strategy and the techniques of the computation. More precise accounts can be found in [Bo2, p. 235/236] or [Bo3, sections 3+4]. It is worth noticing that we only need $E = \mathbf{R}$ as coefficients and S containing no finite places. That is, whenever one reads "S-arithmetic", one may drop the S, and think of the example $\Gamma = SL_n(O_F)$.

Some of the rather basic details skipped in [Bo1] are explained in [Bo5] and [BW] which the speaker may consult when preparing the talk.

Many results are stated for a class of algebraic (or Lie) group satisfying certain properties which hold in particular for SL_n. If possible, the speaker should both define these properties and explain why they are satisfied in the case of SL_n. Furthermore, the example of $SL_n(\mathbf{Z})$ should be thought of as the guiding example, and this example should be brought up whenever possible.

The speakers should keep in mind that some people in the audience are not familliar with the methods from analysis needed in this seminar. A standard reference for de Rham cohomology, harmonic forms etc. is [Wa].

The speakers of the talks 3-5 should work closely together, and so should the speakers of talks 7-9.

References

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